Direct perturbation theory for the dark soliton solution to the nonlinear Schrödinger equation with normal dispersion

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After finding the basic solutions of the linearized nonlinear Schrödinger equation by the method of separation of variables, the perturbation theory for the dark soliton solution is constructed by linear Green's function theory. In application to the self-induced Raman scattering, the adiabatic corrections to the soliton's parameters are obtained and the remaining correction term is given as a pure integral with respect to the continuous spectral parameter.

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I. INTRODUCTION

The nonlinear Schrödinger (NLS) equation supports dark soliton solutions in the case of normal group-velocity dispersion in optical fibers or self-defocusing nonlinearity in waveguides. Dark solitons appear as an intensity dip in the modulationally stable background. They were discovered by Hasegava and Tappert in 1973 [1] and then became a subject of intensive theoretical and experimental studies [2].

An inverse scattering transform (IST) was proposed to solve the NLS equation with normal dispersion (NLS^+ equation for brevity)

$$iu_t - u_{xx} + 2|u|^2 u = 0 \tag{1}$$

under nonvanishing boundary conditions $|u| \to \rho$ as $x \to \pm\infty$, and exact dark-soliton solutions were found [3,4]:

$$u(x,t) = \rho \frac{(\mu + i\nu)^2 + \exp Z}{1 + \exp Z} e^{i2\rho^2 t}, \quad \nu^2 = 1 - \mu^2, \tag{2}$$

where $Z = 2\nu\rho(x - x_0 + \mu\rho t)$. Dark solitons can also be obtained by assuming a solution of the form $u(x,t) = f(x)\exp[i\varphi(x,t)]$ and then solving the ordinary differential equations satisfied by $f(x)$ and $\varphi(x,t)$; see Ref. [5]. A dark soliton solution thus obtained is

$$u(x,t) = \rho \tanh \theta e^{i\varphi}, \tag{3}$$

where

$$\theta = \rho(x - x_0 + Vt), \tag{4}$$

$$\varphi = \left(2\rho^2 + \frac{1}{4}V^2\right)t + \frac{1}{2}Vx + \varphi_0, \tag{5}$$

which is an antisymmetric function of time with an abrupt π phase shift and zero intensity at its center. It is often referred to as the fundamental dark soliton (or black soliton) [4].

In real situations of physical applications of solutions, higher-order effects and other external corrections are often treated as small perturbations. Two systematic approaches of perturbation theory have been established for some typical nonlinear evolution equations since 1970s. They are so-called perturbation theory based on IST [6] and direct perturbation theory [7]. The difficulties in developing perturbation theory for dark solitons of the NLS^+ equation originate from the nonvanishing boundary conditions which may be changed when the corrections present. The perturbation theory based on IST requires an assumption of fixed boundary conditions [8]; therefore, this approach is not appropriate for dark solitons generally. A direct perturbation theory for dark soliton solution (2) was developed based on a complete set of squared Jost solutions [9], the completeness is proved by direct substitution of the explicit expressions of the Jost solutions, and difficulties caused by the background are overcome.

Direct perturbation theory for the solution (3) was also examined [10,11]. However, it is unsatisfactory because the basic solutions referring to the point of the discrete spectrum are obtained by direct observation, which makes trouble for the basic solutions because of the ignorance of the analytic property, which is necessary in the normal analytic continuation procedure [12].

The aim of this paper is to formulate the direct perturbation theory for dark soliton solution (3) clearly and completely. Our approach is developed in the framework of Green's function theory for linear differential equations; it is concise and easy to understand.

This paper is organized as follows. The general expressions of the linearized perturbed NLS^+ equation are given in Sec. II. After introducing an affine parameter to avoid double-valued functions, the linearized equation is solved by the method of separation of variables in Sec. III. The two pairs of independent basic solutions are redefined so that all desired properties of these solutions under asymptotic conditions as well as under the reduction transformation are fulfilled. The orthogonality and completeness of the expanding base are proved by a $1+1$ Green's theorem and Green's function theory in Sec. IV. In application to the self-induced Ramam scattering (SRS) effect [13,14] the adiabatic corrections to the dark soliton's parameters are obtained; the correction due to the continuous parameter is given by a pure integral. The calculations of corrections are shown in Sec. V. Finally, the work is summarized and unreasonable points in the previous papers [10,11] are mentioned in Sec. VI.

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II. PERTURBED NLS$^+$ EQUATION

The perturbed NLS$^+$ equation can be written as

$$iv_t - v_{xx} + 2|v|^2 v = i\epsilon p[v], \tag{6}$$

where ϵ is a small positive parameter and $p[v]$ is a functional of v.

We assume that [7]

$$v = u^a + \epsilon q, \tag{7}$$

where u^a is the adiabatic solution which has the same functional form as the exact dark soliton solution (3), but the parameters involved depend linearly on ϵt, and ϵq is the remain correction term. Substituting Eq. (7) into Eq. (6) yields

$$iq_t - q_{xx} + 4|u|^2 q + 2u^2 \bar{q} = iP[u], \tag{8}$$

where $P[u]$ is the effective source,

$$P[u] = p[u] - s[u], \quad s[u] = \frac{d}{d\tau}u, \tag{9}$$

and $\tau = \epsilon t$ is the slow time in the multitime expansion theory [15]. Since Eq. (9) is of the order of ϵ, u on the left-hand side (LHS) of Eq. (8) is the exact dark soliton.

Substituting the explicit expression of the dark solution, Eq. (3), and introducing

$$e^{-i\varphi} q = g, \quad e^{i\varphi}\bar{q} = \bar{g}, \tag{10}$$

Eq. (8) is written as

$$ig_t - 2\rho^2 g - g_{xx} - iVg_x + 4\rho^2 \tanh^2 \theta g + 2\rho^2 \tanh^2 \theta \bar{g} = ie^{-i\varphi}P. \tag{11}$$

Equation (11) and its complex conjugate can be written in a matrix form

$$\mathcal{L}\boldsymbol{g} = i\widetilde{\boldsymbol{P}}, \quad \widetilde{\boldsymbol{P}} = e^{-i\varphi\sigma_3}\boldsymbol{P}, \quad \boldsymbol{P} = \begin{pmatrix} P \\ \bar{P} \end{pmatrix}, \tag{12}$$

where $\sigma_j, j=1,2,3$, are Pauli matrices and the linearized operator $\mathcal{L}$ is

$$\mathcal{L} = \begin{pmatrix} i\partial_t - \partial_x^2 - iV\partial_x + 4\rho^2 \tanh^2 \theta - 2\rho^2 & 2\rho^2 \tanh^2 \theta \\ -2\rho^2 \tanh^2 \theta & i\partial_t + \partial_x^2 - iV\partial_x - 4\rho^2 \tanh^2 \theta + 2\rho^2 \end{pmatrix}. \tag{13}$$

III. BASIC SOLUTIONS OF THE LINEARIZED EQUATION

A. Separation of variables

The next problem is to find the basic solutions of the linearized operator with zero eigenvalue

$$\mathcal{L}\boldsymbol{W} = 0. \tag{14}$$

We can separate variables by a coordinate transformation to the center of the soliton and return to the originate coordinates after obtaining the solutions. Just by simple calculations,

$$\boldsymbol{W}(t,x)_{\cdot 1} = e^{i4\kappa\lambda t - i2\kappa Vt}e^{-i2\kappa x}\begin{pmatrix} \{\rho \tanh \theta + i\zeta\}^2 \\ \{\rho \tanh \theta - i\rho^2\zeta^{-1}\}^2 \end{pmatrix}, \tag{15}$$

$$\boldsymbol{W}(t,x)_{\cdot 2} = e^{-i4\kappa\lambda t + i2\kappa Vt}e^{i2\kappa x}\begin{pmatrix} \{\rho \tanh \theta + i\rho^2\zeta^{-1}\}^2 \\ \{\rho \tanh \theta - i\zeta\}^2 \end{pmatrix} \tag{16}$$

are two solutions for Eq. (14) with constant factor $e^{\pm i2\kappa x_0}$ dropped. Here κ is real and an affine parameter ζ in introduced to avoid double-value functions of κ such that

$$\kappa = \frac{1}{2}(\zeta - \rho^2\zeta^{-1}), \quad \lambda = \frac{1}{2}(\zeta + \rho^2\zeta^{-1}). \tag{17}$$

Then $\boldsymbol{W}(t,x)_{\cdot 1}$ and $\boldsymbol{W}(t,x)_{\cdot 2}$ satisfy

$$\mathcal{L}\boldsymbol{W}(t,x)_{\cdot 1} = 0, \quad \mathcal{L}\boldsymbol{W}(t,x)_{\cdot 2} = 0. \tag{18}$$

We also have

$$\mathcal{L}\boldsymbol{W}(x)_{\cdot 1} = 2\kappa(2\lambda - V)\boldsymbol{W}(x)_{\cdot 1},$$
$$\mathcal{L}\boldsymbol{W}(x)_{\cdot 2} = -2\kappa(2\lambda - V)\boldsymbol{W}(x)_{\cdot 2}, \tag{19}$$

where $\boldsymbol{W}(x)_{\cdot 1}$ is $\boldsymbol{W}(t,x)_{\cdot 1}$ without $e^{i4\kappa\lambda t - i2\kappa Vt}$, for example.

B. Desired properties of the basic solutions

In consideration of various properties we define four solutions as follows:

$$\widetilde{\Psi}(t,x,\zeta) = e^{i4\kappa\lambda t - i2\kappa Vt}e^{-i2\kappa x}\begin{pmatrix} \{\rho \tanh \theta + i\zeta\}^2 \\ \{\rho \tanh \theta - i\rho^2\zeta^{-1}\}^2 \end{pmatrix}(\rho + i\zeta)^{-2}, \tag{20}$$

$$\Psi(t,x,\zeta) = e^{-i4\kappa\lambda t + i2\kappa Vt}e^{i2\kappa x}\begin{pmatrix} \{\rho \tanh \theta + i\rho^2\zeta^{-1}\}^2 \\ \{\rho \tanh \theta - i\zeta\}^2 \end{pmatrix}(\rho - i\zeta)^{-2}, \tag{21}$$

$$\Phi(t,x,\zeta) = e^{i4\kappa\lambda t - i2\kappa Vt}e^{-i2\kappa x}\begin{pmatrix} \{\rho \tanh \theta + i\zeta\}^2 \\ \{\rho \tanh \theta - i\rho^2\zeta^{-1}\}^2 \end{pmatrix}(\rho - i\zeta)^{-2}, \tag{22}$$

$$\tilde{\Phi}(t,x,\zeta) = e^{-i4\kappa\lambda t + i2\kappa Vt} e^{i2\kappa x} \begin{pmatrix} \{\rho \tanh \theta + i\rho^2 \zeta^{-1}\}^2 \\ \{\rho \tanh \theta - i\zeta\}^2 \end{pmatrix} (\rho + i\zeta)^{-2}. \tag{23}$$

They have the following properties: in the complex ζ plane excluding 0 and ∞, $\Phi(t,x,\zeta)$ and $\Psi(t,x,\zeta)$ have a double pole at $\zeta=-i\rho$ and $\tilde{\Phi}(t,x,\zeta)$ and $\tilde{\Psi}(t,x,\zeta)$ have a double pole at $\zeta=i\rho$; elsewhere, they are analytic. In the limit $x\to\infty$ we have

$$\tilde{\Psi}(x,\zeta) \to e^{-i2\kappa x} \begin{pmatrix} 1 \\ -\rho^2\zeta^{-2} \end{pmatrix}, \tag{24}$$

$$\Psi(x,\zeta) \to e^{i2\kappa x} \begin{pmatrix} -\rho^2\zeta^{-2} \\ 1 \end{pmatrix}, \tag{25}$$

and in the limit $x\to-\infty$

$$\Phi(x,\zeta) \to e^{-i2\kappa x} \begin{pmatrix} 1 \\ -\rho^2\zeta^{-2} \end{pmatrix}, \tag{26}$$

$$\tilde{\Phi}(x,\zeta) \to e^{i2\kappa x} \begin{pmatrix} -\rho^2\zeta^{-2} \\ 1 \end{pmatrix}. \tag{27}$$

We also have the properties

$$\Phi(t,x,\zeta) = a(\zeta)^2 \tilde{\Psi}(t,x,\zeta), \quad a(\zeta) = \frac{\zeta - i\rho}{\zeta + i\rho}, \tag{28}$$

$$\tilde{\Phi}(t,x,\zeta) = \tilde{a}(\zeta)^2 \Psi(t,x,\zeta), \quad \tilde{a}(\zeta) = \frac{\zeta + i\rho}{\zeta - i\rho}. \tag{29}$$

Moreover, under the transformation $\zeta\to\eta=\rho^2\zeta^{-1}$, we have

$$\tilde{\Psi}(x,\zeta) = -\rho^{-2}\eta^2 \Psi(x,\eta), \tag{30}$$

$$\tilde{\Phi}^A(x,\zeta) = -\rho^{-2}\eta^2 \Psi(x,\eta). \tag{31}$$

IV. ORTHOGONALITY AND COMPLETENESS

A. Adjoint operator

We now introduce the adjoint operator by partial integration. Consider

$$\boldsymbol{g}^T \mathcal{L} \boldsymbol{f} - \boldsymbol{f}^T \mathcal{L}^A \boldsymbol{g} = \text{divergence}, \tag{32}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are rays with two components. The adjoint operator is, obviously,

$$\mathcal{L}^A = \begin{pmatrix} -i\partial_t - \partial_x^2 + iV\partial_x + 4\rho^2 \tanh^2 \theta - 2\rho^2 & -2\rho^2 \tanh^2 \theta \\ 2\rho^2 \tanh^2 \theta & -i\partial_t + \partial_x^2 + iV\partial_x - 4\rho^2 \tanh^2 \theta + 2\rho^2 \end{pmatrix}, \tag{33}$$

and the divergence term is

$$i\partial_t(f_1g_1 + f_2g_2) - iV\partial_x(f_1g_1 + f_2g_2) - \partial_x(f_{1x}g_1 - f_1g_{1x} - f_{2x}g_2 + f_2g_{2x}). \tag{34}$$

Comparison of Eqs. (13) and (33) gives

$$-\sigma_2 \mathcal{L} \sigma_2 = \mathcal{L}^A; \tag{35}$$

this indicates that the basic solution of the adjoint operator is

$$\mathcal{L}^A W^A = 0, \quad W^A = i\sigma_2 W. \tag{36}$$

Since in terms of ζ two values of ζ correspond to one value of κ or λ via Eq. (17), we must restrict the domain of ζ—for example, $|\zeta|>\rho$. In this restriction, the independent basic solutions of the linearized equation may be chosen as $\tilde{\Psi}(t,x,\zeta)$ and $\Psi(t,x,\zeta)$, because they have different asymptotic behaviors. Then the independent adjoint solutions are $\Phi^A(t,x,\zeta)$ and $\tilde{\Phi}^A(t,x,\zeta)$.

B. (1+1) Green's theorem and orthogonality

Assuming that $\boldsymbol{f}=\Psi(x,t,\zeta)$ in Eq. (32) is the basic solution to the linearized operator $\mathcal{L}$ and $\boldsymbol{g}=\Phi^A(x,t,\zeta')$ to $\mathcal{L}^A$, then the LHS of Eq. (32) equals zero. Integration of the divergence term on the RHS over $(-L,L)$ for x and $(0,t)$ for t yields

$$i\int_{-L}^{L} dx(f_1g_1 + f_2g_2)|_{t=0}^{t=t} = \int_0^t dt\{iV(f_1g_1 + f_2g_2) + (f_{1x}g_1 - f_1g_{1x} - f_{2x}g_2 + f_2g_{2x})\}|_{x=-L}^{x=L}. \tag{37}$$

The time factor on the RHS is simply $e^{i4(\kappa\lambda-\kappa'\lambda')t-i2(\kappa-\kappa')Vt}$, since $|x|\to\infty$. Integration with t is given a factor $\frac{1}{i4(\kappa\lambda-\kappa'\lambda')-i2(\kappa-\kappa')V}$. Thus Eq. (37) reduces to

$$i\int_{-L}^{L} dx(f_1g_1 + f_2g_2) = \frac{1}{i4(\kappa\lambda - \kappa'\lambda') - i2(\kappa - \kappa')V} \times \{iV(f_1g_1 + f_2g_2) + (f_{1x}g_1 - f_1g_{1x} - f_{2x}g_2 + f_2g_{2x})\}|_{x=-L}^{x=L}, \tag{38}$$

where $t=0$. It is easy to know that $\boldsymbol{f}_x=i2\kappa\boldsymbol{f}$, $\boldsymbol{g}_x=-i2\kappa'\boldsymbol{g}$, and the brackets on the RHS are equal to $iV(f_1g_1+f_2g_2)+i2(\kappa+\kappa')(f_1g_1-f_2g_2)$. Noting that in

$$i4(\kappa\lambda-\kappa'\lambda')-i2(\kappa-\kappa')V = i(\zeta-\zeta')[(\zeta+\zeta')(1+\rho^4\zeta^{-2}\zeta'^{-2})-V(1+\rho^2\zeta^{-1}\zeta'^{-1})] \quad (39)$$

and

$$i2(\kappa+\kappa')(1+\rho^4\zeta^{-2}\zeta'^{-2})+iV(\rho^4\zeta^{-2}\zeta'^{-2}-1) = i(1-\rho^2\zeta^{-1}\zeta'^{-1})[(\zeta+\zeta')(1+\rho^4\zeta^{-2}\zeta'^{-2}) - V(1+\rho^2\zeta^{-1}\zeta'^{-1})] \quad (40)$$

the squared brackets of them are the same, Eq. (38) thus becomes

$$\int_{-L}^{L} dx(f_1g_1+f_2g_2)=\frac{(1-\rho^2\zeta^{-1}\zeta'^{-1})}{i(\zeta-\zeta')}[a(\zeta')^2e^{i2(\kappa-\kappa')L} - a(\zeta)^2e^{-i2(\kappa-\kappa')L}]. \quad (41)$$

Here we have used Eqs. (24) and (26).

Since $|\zeta|,|\zeta'|>\rho$ and $2(\kappa-\kappa')=(\zeta-\zeta')(1+\rho^2\zeta^{-1}\zeta'^{-1})$, we have

$$\lim_{L\to\infty}\frac{1}{i(\zeta-\zeta')}e^{i2(\kappa-\kappa')L}=\pi\delta(\zeta-\zeta'). \quad (42)$$

As $L\to\infty$ Eq. (41) becomes

$$\langle\Phi(\zeta')|\Psi(\zeta)\rangle = 2\pi a(\zeta)^2(1-\rho^2\zeta^{-2})\delta(\zeta-\zeta'), \quad (43)$$

where $\lim_{L\to\infty}\int_{-L}^{L}dx(f_1g_1+f_2g_2)=\langle\Phi(\zeta')|\Psi(\zeta)\rangle$. Similarly,

$$\langle\tilde{\Phi}(\zeta')|\tilde{\Psi}(\zeta)\rangle = -2\pi\tilde{a}(\zeta)^2(1-\rho^2\zeta^{-2})\delta(\zeta-\zeta'). \quad (44)$$

C. Choice of ζ in the whole range $\{-\infty,\infty\}$

Since functions are written in terms of an affine parameter ζ, we should consider their properties under the reduction transformation $\zeta\to\rho^2\zeta^{-1}$. Equation (32) leads to

$$\delta(\zeta-\zeta')=\delta(\rho^2\eta^{-1}-\rho^2\eta'^{-1})=\frac{\eta\eta'}{\rho^2}\delta(\eta-\eta'), \quad (45)$$

where $\zeta=\rho^2\eta^{-1}$, $\zeta'=\rho^2\eta'^{-1}$. Therefore, substituting Eqs. (30) and (45) into Eq. (44) leads to

$$\langle\Phi(\eta')|\Psi(\eta)\rangle\rho^{-4}\eta'^2\eta^2 = -2\pi a(\eta)^2(1-\rho^{-2}\eta^2)\frac{\eta\eta'}{\rho^2} \times\delta(\eta-\eta'). \quad (46)$$

Canceling the same factors on the two sides we obtain

$$\langle\Phi(\eta')|\Psi(\eta)\rangle = 2\pi a(\eta)^2(1-\rho^2\eta^{-2})\delta(\eta-\eta'), \quad (47)$$

with the same form as Eq. (43), but $|\eta|,|\eta'|<\rho$. Thus Eq. (43) is valid for ζ and ζ' in the whole range $\{-\infty,\infty\}$. Therefore, the independent basic solution is only 1, and we choose $\Psi(t,x,\zeta)$. The corresponding adjoint solution is $\Phi^A(t,x,\zeta)$.

When ζ, the argument of $\Psi(x,\zeta)$, runs from $-\infty$ to ∞, it can be analytically continued into the upper half plane of complex ζ. Thus from

$$\mathcal{L}\Psi(t,x,\zeta)=0, \quad \mathcal{L}\Psi(x,\zeta)=-(4\kappa\lambda-2\kappa V)\Psi(x,\zeta), \quad (48)$$

we have

$$\mathcal{L}\Psi(t,x,i\rho)=0, \quad \mathcal{L}\Psi(x,i\rho)=i2\rho V\Psi(x,i\rho) \quad (49)$$

and

$$\mathcal{L}\dot{\Psi}(t,x,i\rho)=0,$$

$$\mathcal{L}\dot{\Psi}(x,i\rho)=i2\rho V\dot{\Psi}(x,i\rho)-i4\rho\Psi(x,i\rho), \quad (50)$$

where $\dot{\Psi}(x,t,i\rho)=\frac{d}{d\zeta}\Psi(x,t,\zeta)|_{\zeta=i\rho}$, since $\mathcal{L}$ does not involve the parameter ζ. Here we notice that $a(\zeta)$ has only one zero $\zeta=i\rho$.

Writing Eq. (41) in the form

$$i(\zeta-\zeta')\int_{-L}^{L}dx(f_1g_1+f_2g_2) = (1-\rho^2\zeta^{-1}\zeta'^{-1}) \times[a(\zeta')^2e^{i2(\kappa-\kappa')L}-a(\zeta)^2e^{-i2(\kappa-\kappa')L}], \quad (51)$$

and then applying the operators $\frac{d}{d\zeta}|_{\zeta=\zeta'=i\rho}$, $\frac{d^2}{d\zeta^2}|_{\zeta=\zeta'=i\rho}$, and $\left(\frac{d^3}{d\zeta^3}+3\frac{d}{d\zeta'}\frac{d^2}{d\zeta^2}\right)|_{\zeta=\zeta'=i\rho}$ to Eq. (51), respectively, we obtain

$$\langle\Phi(i\rho)|\Psi(i\rho)\rangle=0,$$

$$\langle\dot{\Phi}(i\rho)|\Psi(i\rho)\rangle=\langle\Phi(i\rho)|\dot{\Psi}(i\rho)\rangle=i2\dot{a}(i\rho)^2,$$

$$\langle\dot{\Phi}(i\rho)|\dot{\Psi}(i\rho)\rangle=i2\dot{a}(i\rho)\ddot{a}(i\rho)-2\rho^{-1}\dot{a}(i\rho)^2, \quad (52)$$

where $i\rho$ is in the upper complex ζ plane.

From Eq. (21) we have

$$\Psi(x,i\rho)=e^{-2\rho x}\frac{1}{4}(\tanh\theta+1)^2\begin{pmatrix}1\\1\end{pmatrix}, \quad (53)$$

$$\dot{\Psi}(x,i\rho)=\Psi(x,i\rho)\frac{i}{\rho}+e^{-2\rho x}\frac{1}{2}(\tanh\theta+1)\begin{pmatrix}1\\-1\end{pmatrix}\frac{i}{\rho}. \quad (54)$$

D. Green's function

The linearized equation is essentially a linear equation, so we can choose an appropriate method of linear equations. The RHS of the linearized equation (12) is not zero; its Green's function $G(x,t,x',t')$ is defined as

$$\mathcal{L}G=\delta(x-x')\delta(t-t'). \quad (55)$$

It is a 2×2 matrix since the basic solution has two components. The solution of Eq. (12) can be written as

$$\boldsymbol{q}=\int_0^\infty dt'\int_{-\infty}^{\infty}dx'G(x,t,x',t')\tilde{\boldsymbol{P}}(x',t'). \quad (56)$$

Because of the causality condition, G should be zero for $t'>t$, hence

$$G = G^0(x,t,x',t')\theta(t-t'), \tag{57}$$

where $\theta(\tau)$ represents the step function—i.e., $\theta(\tau)=1$ for $\tau > 0$ and $\theta(\tau)=0$ for $\tau < 0$. Then Eq. (56) becomes

$$\boldsymbol{q} = \int_0^t dt' \int_{-\infty}^{\infty} dx' G^0(x,t,x',t')\widetilde{\boldsymbol{P}}(x',t'), \tag{58}$$

and this solution fulfills the initial condition $q=0$ for $t=0$. Inserting Eq. (58) into Eq. (12) gives

$$\int_{-\infty}^{\infty} dx' G^0(x,t,x',t)\widetilde{\boldsymbol{P}}(x',t) + \int_0^t dt' \int_{-\infty}^{\infty} dx' \mathcal{L} G^0(x,t,x',t')\widetilde{\boldsymbol{P}}(x',t') = \widetilde{\boldsymbol{P}}(x,t). \tag{59}$$

This equation is identically satisfied by assuming that G^0 is a solution of the homogeneous equation

$$\mathcal{L} G^0(x,t,x',t') = 0 \tag{60}$$

and obeys the final condition

$$G^0\,(x,t,x',t')|_{t=t'} = \delta(x-x'), \tag{61}$$

noticing that these are all 2×2 matrices. The Green's function G^0 is completely determined by the above two conditions.

E. Proof the completeness

The functions $\Psi(x,t,\zeta)$, $\Psi(x,t,i\rho)$, and $\dot{\Psi}(x,t,i\rho)$ are the basic solutions of the linearized equation—i.e., solutions of homogeneous equation. Moreover, from the properties of the reduction transformation, the state labeled with a tilde is not needed.

The above three are the basic solutions of the linearized equation, so we have

$$G^0(x,t,x',t') = \int_{-\infty}^{\infty} d\zeta\, \Psi(x,t,\zeta)A(x',t',\zeta) + \Psi(x,t,i\rho)B(x',t',i\rho) + \dot{\Psi}(x,t,i\rho)C(x',t',i\rho), \tag{62}$$

where A, B, and C are undetermined 1×2 matrices. Equation (61) then becomes

$$\delta(x-x') = \int_{-\infty}^{\infty} d\zeta\, \Psi(x,t,\zeta)A(x',t,\zeta) + \Psi(x,t,i\rho)B(x',t,i\rho) + \dot{\Psi}(x,t,i\rho)C(x',t,i\rho).$$

Multiplying Eq. (63) by $\Phi^A(x,t,\zeta')^T$ from the left and integrating over x yields

$$\Phi^A(x',t,\zeta')^T = \int d\zeta\, M(t,\zeta',\zeta)A(x',t,\zeta), \tag{63}$$

where

$$M(t,\zeta',\zeta) = \int_{-\infty}^{\infty} dx\, \Phi^A(x,t,\zeta')^T\Psi(x,t,\zeta) = 2\pi a(\zeta)^2(1-\rho^2\zeta^{-2})\delta(\zeta-\zeta'), \tag{64}$$

the summary terms vanish for the orthogonality, and the final expression of M comes from Eq. (43). Then from Eqs. (63) and (64) we have

$$A(x',t,\zeta) = \frac{1}{2\pi a(\zeta)^2(1-\rho^2\zeta^{-2})}\Phi^A(x',t,\zeta)^T. \tag{65}$$

Similarly, multiplying Eq. (63) by $\Phi^A(x,t,i\rho)^T$ and $\dot{\Phi}^A(x,t,i\rho)^T$ from the left, respectively, and integrating over x, after some work we also obtain the expressions of B and C. Substituting the expressions of A, B, and C into Eq. (63), we obtain the equation of completeness,

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\zeta \frac{1}{a(\zeta)^2(1-\rho^2\zeta^{-2})}\Psi(x,\zeta)\Phi^A(x',\zeta)^T - i\frac{1}{2\dot{a}(i\rho)^2}[\dot{\Psi}(x,i\rho)\Phi^A(x',i\rho)^T + \Psi(x,i\rho)\dot{\Phi}^A(x',i\rho)^T] + i\left(\frac{\ddot{a}(i\rho)}{2\dot{a}(i\rho)^3} + \frac{2\rho^2 i\rho^{-3}}{4\dot{a}(i\rho)^2}\right)\Psi(x,i\rho)\Phi^A(x',i\rho)^T. \tag{66}$$

V. CALCULATIONS OF THE FIRST-ORDER APPROXIMATION

A. Secularity conditions

According to the completeness equation (66) the unknown function $|g\rangle$ can be expanded as

$$|g\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\zeta\, g(\zeta)|\Psi(\zeta)\rangle + g_d|\Psi(i\rho)\rangle + f_d|\dot{\Psi}(i\rho)\rangle, \tag{67}$$

where $g(\zeta)$, g_d, and f_d are functions of t. Applying the operator $\mathcal{L}$ and taking account of Eq. (12) yields

$$i|\widetilde{\boldsymbol{P}}\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\zeta\{ig_t - g(4\kappa\lambda - 2\kappa V)\}|\Psi(\zeta)\rangle + \{ig_{dt} + (i2\rho V)g_d\}|\Psi(i\rho)\rangle + \{if_{dt} + (i2\rho V)f_d\}|\dot{\Psi}(i\rho)\rangle - (i4\rho)f_d|\Psi(i\rho)\rangle. \tag{68}$$

Multiplying $\langle\Phi(i\rho)|$ from the left of Eq. (68), we obtain

$$[if_{dt} + (i2\rho V)f_d]i2\dot{a}(i\rho)^2 = i\langle\Phi(i\rho)|\widetilde{\boldsymbol{P}}\rangle, \tag{69}$$

so that f_d will tend to infinity as $t\to\infty$, unless

$$\langle\Phi(i\rho)|\widetilde{\boldsymbol{P}}\rangle = 0. \tag{70}$$

Similarly, we demand

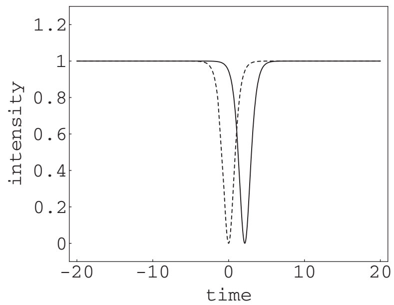

FIG. 1. Shape of a fundamental dark soliton after propagation distance $z=40$ with parameter $\tau_d=0.01$. The dashed curve is the initial pulse.

$$\langle\dot{\Phi}(i\rho)|\tilde{\boldsymbol{P}}\rangle=0. \tag{71}$$

These two conditions, Eqs. (70) and (71), are called security conditions.

B. Application: Self-induced Raman scattering

As an example of the application of the perturbation theory developed above, we consider the dynamics of dark solitons under the influence of self-induced Raman scattering. It is described by the NLS$^+$ equation with the SRS term $\tau_d u(|u|^2)_t$,

$$iu_z-\frac{1}{2}u_{tt}+|u|^2u=\tau_d u(|u|^2)_t; \tag{72}$$

here, z is the spatial variable, t is the temporal variable, and τ_d is the normalized time delay. The SRS item, which is one of the important perturbations acting on dark solitons in fibers, appears when the pulse duration in fibers reaches the subpicosecond regime. It was investigated both numerically and analytically [13,14].

Here we rewrite Eq. (72) in our style by $z\to 2t$ and $t\to x$,

$$iu_t-u_{xx}+2|u|^2u=\epsilon u\frac{\partial}{\partial x}(|u|^2). \tag{73}$$

The adiabatic soliton solution for Eq. (73) is Eq. (3) with the difference that parameters are not written that strictly, but

$$\theta=\rho(x-\xi),\quad \varphi=\frac{1}{2}V(x-\xi)+\delta. \tag{74}$$

In the unperturbed case, we have

$$\xi_t=-V,\quad V_t=0,\quad \rho_t=0,\quad \delta_t=-\frac{V^2}{4}+2\rho^2. \tag{75}$$

From Eq. (22) we obtain

$$\Phi^A(x,i\rho)=\frac{e^{2\rho x}}{4}(\tanh\theta-1)^2\begin{pmatrix}1\\-1\end{pmatrix}, \tag{76}$$

$$\dot{\Phi}^A(x,i\rho)=\frac{i}{\rho}\Phi+\frac{e^{2\rho x}}{2}\frac{i}{\rho}(\tanh\theta-1)\begin{pmatrix}-1\\-1\end{pmatrix}. \tag{77}$$

Hence the secularity conditions, Eqs. (70) and (71), reduce to

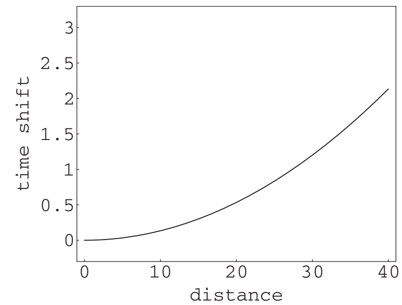

FIG. 2. Trace of the soliton as a function of propagation distance.

$$\int_{-\infty}^{\infty}d\theta\, e^{2\theta}(\tanh\theta-1)^2\,\mathrm{Im}(pe^{-i\varphi})=\int_{-\infty}^{\infty}d\theta\, e^{2\theta}(\tanh\theta-1)^2\,\mathrm{Im}(u_\tau e^{-i\varphi}) \tag{78}$$

and

$$\int_{-\infty}^{\infty}d\theta\, e^{2\theta}(\tanh\theta-1)\mathrm{Re}(pe^{-i\varphi})=\int_{-\infty}^{\infty}d\theta\, e^{2\theta}(\tanh\theta-1)\mathrm{Re}(u_\tau e^{-i\varphi}) \tag{79}$$

by changing the integral variable $x\to\theta=\rho(x-\xi)$. In Eq. (79), the terms $\Phi^A(x,i\rho)$ in the RHS of Eq. (77) were canceled in both sides on account of Eq. (76).

The SRS term gives

$$e^{-i\varphi}p=(-i)e^{-i\varphi}u(|u|^2)_x=2(-i)\rho^4\tanh^2\theta\,\mathrm{sech}^2\theta, \tag{80}$$

by substituting Eq. (12). The LHSs of Eqs. (78) and (79) are $-8\rho^2/15$ and 0, respectively.

Since

$$u_\tau=\rho_\tau\frac{d}{d\rho}u+V_\tau\frac{d}{dV}u+\xi_\tau\frac{d}{d\xi}u+\delta_\tau\frac{d}{d\delta}u, \tag{81}$$

we have

$$e^{-i\varphi}u_\tau=\rho_\tau(\tanh\theta+\theta\,\mathrm{sech}^2\theta)+\xi_\tau\left(-\rho^2\,\mathrm{sech}^2\theta-\frac{i\rho V}{2}\tanh\theta\right)+V_\tau\frac{i}{2}\theta\tanh\theta+\delta_\tau i\rho\tanh\theta. \tag{82}$$

The RHS of Eq. (78) is $V_\tau/2$, and that of Eq. (79) is

$$2\rho^2\xi_\tau+\int_{-\infty}^{\infty}d\theta\rho_\tau(-\tanh^2\theta-\theta\sinh\theta\,\mathrm{sech}^3\theta). \tag{83}$$

Then from Eqs. (78) and (79) we obtain the evolution for the SRS effect:

$$V_\tau=-\frac{16}{15}\rho^4,\quad \xi_\tau=0,\quad \rho_\tau=0. \tag{84}$$

Taking account of Eq. (75), the evolution of the adiabatic solution parameter, the formulas are obtained up to ϵ:

$$\xi_t = -V, \quad V_t = -\frac{16}{15}\rho^4\epsilon, \quad \rho_t = 0. \tag{85}$$

These derive

$$\rho = \rho_0, \quad V = -\frac{16}{15}\rho^4\epsilon t + V_0 \tag{86}$$

and

$$\xi = \frac{8}{15}\rho^4\epsilon t^2 - V_0 t + x_0. \tag{87}$$

Taking into consideration that our spatial variable t is half of the one used in the literature—i.e., $2t=z$—the time shift we obtained is $2\rho^4\tau_d z^2/15$, which is in agreement with the numerical simulation by Zhao and Bourkoff [13]. Figure 1 shows the pulse shape after a fundamental dark soliton propagates a distance $z=40$ with normalized time delay $\tau_d = 0.01$, and Fig. 2 gives the trace of the soliton as a function of propagation distance z.

C. Continuous spectrum

Multiplying $\langle\Phi(\zeta)|$ from the left of Eq. (68), we obtain an equation for the continuous spectrum part:

$$\{ig_t - 2\kappa(2\lambda - V)g\}a^2(\zeta)(1-\rho^2\zeta^{-2}) = i\langle\Phi(\zeta)|\widetilde{\boldsymbol{P}}\rangle, \tag{88}$$

which is equivalent to

$$i\partial_{t'}\{ge^{i2\kappa'(2\lambda'-V)t'}\}a^2(\zeta)(1-\rho^2\zeta^{-2}) = e^{i2\kappa'(2\lambda'-V)t'}i\langle\Phi(\zeta)|\widetilde{\boldsymbol{P}}\rangle, \tag{89}$$

so that

$$g(\zeta,t) = \frac{1}{a^2(\zeta)(1-\rho^2\zeta^{-2})}\int_0^t dt'\, e^{2i\kappa(2\lambda-V)(t'-t)}\langle\Phi(\zeta)|\widetilde{\boldsymbol{P}}(t')\rangle. \tag{90}$$

Since Eq. (88) is of the order of ϵ and the factor ϵ is canceled to sides so that $\langle\Phi(\zeta)|\widetilde{\boldsymbol{P}}\rangle$ do not involve terms of order ϵ, the term $\widetilde{\boldsymbol{P}}$ thus reduces to

$$\widetilde{\boldsymbol{p}} = e^{-i\varphi\sigma_3}\boldsymbol{p} = \begin{pmatrix} e^{-i\varphi}p \\ e^{i\varphi}\bar{p} \end{pmatrix}. \tag{91}$$

For the SRS effect, $pe^{-i\varphi}$ is given in Eq. (80). Noting Eqs. (22) and (36), we have

$$\Phi^A(t,x,\zeta) = e^{i2\kappa(2\lambda-V)t}e^{-i2\kappa x}\begin{pmatrix} \{\rho\tanh\theta + i\zeta\}^2 \\ -\{\rho\tanh\theta - i\rho^2\zeta^{-1}\}^2 \end{pmatrix}(\rho - i\zeta)^{-2}. \tag{92}$$

Hence it is given as

$$\langle\Phi(\zeta)|\widetilde{\boldsymbol{P}}\rangle = e^{i2\kappa(2\lambda-V)t}\int_{-\infty}^{\infty} dx\, i(\rho-i\zeta)^{-2}e^{-i2\kappa x}2\rho^4\tanh^2\theta\,\mathrm{sech}^2\theta \times\{(\rho\tanh\theta+i\zeta)^2 + (\rho\tanh\theta - i\rho^2\zeta^{-1})^2\}. \tag{93}$$

Since $dx=\frac{1}{\rho}d\theta$, substituting $A=\frac{-2\kappa}{\rho}$, the integral in Eq. (93) can be evaluated and Eq. (90) is given by

$$g(\zeta,t) = \frac{(-\rho^5)I}{a^2(\zeta)(1-\rho^2\zeta^{-2})(\rho-i\zeta)^2(2\lambda-V)} \times[e^{2i\kappa(2\lambda t-x_0)} - e^{2i\kappa(Vt-x_0)}]; \tag{94}$$

here, I is a complex number coming from the integral over θ employing the residue theorem,

$$I = \left[2\rho^2\left(\frac{A^5}{120} - \frac{A^3}{6} + \frac{A}{6}\right) + (2\rho^3\zeta^{-1} - 2\rho\zeta)\left(-\frac{A^4}{24} + \frac{A^2}{3}\right) - (\zeta^2 + \rho^4\zeta^{-2})\left(-\frac{A^3}{6} + \frac{A}{3}\right)\right]\pi\,\mathrm{csch}\,\frac{\pi A}{2}. \tag{95}$$

VI. SUMMARY AND DISCUSSION

In summary, we have developed a direct perturbation theory for dark solutions to the NLS$^+$ equation, which started from the construction of basic solutions of the linearized equation. The basic solutions related to the continuous spectrum were found in solving the equation by separating variables, and those related to the discrete spectrum were obtained by analytic continuation from them. Also, perturbation theory was applied to the SRS effect.

The same problem was discussed in previous work [10,11]. In their work, basic solutions relating to the points of discrete spectrum were obtained by direct observation. It gave the solutions trouble resulting from ignorance of the analytic property, which is necessary in the normal analytic continuation procedure.

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